Topics in complex analysis. Lecture 5 (Oct 14, 2024)

The order $o_c(f) \in \mathbb{N} \cup \{0, \infty\}$ is

- 0 if c is not a zero of f (ie $f(c) \neq 0$)
- ∞ if f vanishes on a nbhd of c
- otherwise: uniquely det by the existence of a holom. fct g def. on a nbhd of c with $g(z) \neq 0$ & $$f(z) = g(z)\,(z-c)^{o_c(f)}$$ $\forall z$ in an appr. nbhd of c.

(equiv c is a zero of $f, f', \dots, f^{(o_c(f)-1)}$)

Proof of Lemma 3.11 Let $c \in \mathcal{U}$. By locally unif cvg we know $\prod_{j\in\mathbb{N}} f_j(c)$ converges in the sense of Def 3.1

Hence $\exists j_0 \in \mathbb{N}$ $f_j(c) \neq 0$ $\forall j \geq j_0$. Write

$$f = f_1 \cdots f_{j_0-1} \underbrace{\overbrace{\prod_{j=j_0}^{\infty} f_j}^{g}}_{\text{loc unif to a nowhere vanishing fct}}$$

We know: g is holo by Cor 3.7.

Hence since $g(c) \neq 0$, c is a zero of f iff it is a zero of f_1 or f_2 or $\cdots$ or f_{j_0-1}. Hence

$$o_c(f) = \sum_{j=1}^{j_0-1} o_c(f_j) = \sum_{j=1}^{\infty} o_c(f_j).$$

Analogously this observation shows $Z(f) \subseteq \bigcup_{j=1}^{\infty} Z(f_j)$

The converse holds trivially. $\square$

Heuristics for the logarithmic derivative f'/f of a holo function $f: U \to \mathbb{C}$, $f \not\equiv 0$

- If f would take values in $(0,\infty)$ then
  $f'/f = (\log f)'$.
- If $f = \prod_{j=1}^{\infty} f_j$ then

$$\frac{f'}{f} = (\log f)' = \left(\sum_{j=1}^{\infty} \log f_j \right)' = \sum_{j=1}^{\infty} (\log f_j)' = \sum_{j=1}^{\infty} \frac{f_j'}{f_j}$$

Proof of Prop 3.12. All functions $e_j = f_j'/f_j$ are holo on $U \setminus Z(f)$. Moreover we write

$$f = f_1 \cdots f_{n-1} \underbrace{\prod_{j=n}^{\infty} f_j}_{g_n} \qquad (*)$$

where g_n holo (cf. Cor 3.7)
and $g_n(z) \neq 0$ $\forall z \in U \setminus Z(f)$. Then by Leibniz rule

$$f' = f_1' f_2 \cdots f_{n-1} g_n + f_1 f_2' f_3 \cdots f_{n-1} g_n + \dots + f_1 f_2 \cdots f_{n-1} g_n'$$

Dividing by $(*)$ yields $\boxed{\dfrac{f'}{f} = \dfrac{f_1'}{f_1} + \dfrac{f_2'}{f_2} + \dots + \dfrac{g_n'}{g_n}}$.

We will show $g_n'/g_n \to 0$ loc unif.

We know by Lemma 3.9 & Cor 3.7 $g_n \to 1$ loc unif. By conv. props of derivatives (Thm 1.5) we know $g_n' \to 0$ loc. unif. Hence

$$g_n'/g_n \longrightarrow 0/1 = 0 \quad \text{loc unif (later)}$$

To tie up loose ends:

Lemma Assume $f_j, g_j : U \longrightarrow \mathbb{C}$ continuous defined on an open set $U \subseteq \mathbb{C}$ converging loc unif to f, g resp. And suppose g never vanishes. Then

$$\frac{f_j}{g_j} \longrightarrow \frac{f}{g} \text{ loc unif.}$$

Proof First of all note f, g continuous (since limits of loc unif cvg and continuous fcts are convergent). Let $K \subseteq U$ cpt subset. and let $\varepsilon > 0$.

want to show: $\exists j_0 \ \forall j \geq j_0$

$$\sup_{z \in K} \left| \frac{f_j}{g_j}(z) - \frac{f}{g}(z) \right| \leq \varepsilon$$

Since g is continuous with g not vanishing on K, we have

$$c := \inf_{z \in K} |g(z)| > 0.$$

Since $g_j \longrightarrow g$ unif on K (Rem 1.2) $\exists j_0 \in \mathbb{N}$ $\forall j \geq j_0$

$$\inf_{z \in K} |g_j(z)| \geq c/2.$$

Now we estimate

Hence
$$\frac{f'}{f} = \lim_{n\to\infty} \sum_{j=1}^{n-1} \frac{f_j'}{f_j} = \sum_{j=1}^{\infty} \frac{f_j'}{f_j} \quad \text{loc unif.}$$

We still need to upgrade this to loc normal cvg. on $U \setminus Z(f)$. Fix $z_0 \in U \setminus Z(f)$ and $r > 0$ st. $\overline{B}_{2r}(z_0) \subseteq U \setminus Z(f)$. Applying Cor 3.7 $f_j \longrightarrow 1$ loc unif to 1 on U. Hence $\exists j_0 \in \mathbb{N}$ st $\forall j \geq j_0$:
$$\inf_{z \in B_r(z_0)} |f_j(z)| \geq \frac{1}{2}.$$

Set $a_j = f_j - 1$. Then want to show finiteness of
$$\sum_{j=j_0}^{\infty} \sup_{z \in B_r(z_0)} \left| \frac{f_j'(z)}{f_j(z)} \right| \leq 2 \sum_{j=j_0}^{\infty} \sup_{z \in B_r(z_0)} |a_j'(z)|$$
$$\overset{(*)}{\leq} 4 \sum_{j=j_0}^{\infty} \frac{1}{r} \sup_{z \in \partial B_{2r}(z_0)} |a_j(z)| < \infty$$

(loc norm. cvg of $\sum_j a_j$

To verify $(*)$ we simply observe by Cor 0.3 $\forall z \in B_r(z_0)$:
$$|a_j'(z)| = \left| \frac{1}{2\pi i} \int_{\partial B_{2r}(z_0)} \frac{a_j(y)}{(y-z)^2} \, dy \right|$$
$$\leq \frac{1}{2\pi} \int_{\partial B_{2r}(z_0)} \frac{|a_j(y)|}{(y-z)^2} \, dy$$
$$\leq \frac{1}{2\pi} \sup_{z \in \partial B_{2r}(z_0)} |a_j(z)| \frac{1}{r^2} \underbrace{\operatorname{Len}(\partial B_{2r}(z_0))}_{4\pi r}$$
$$= 2 \frac{1}{r} \sup_{z \in \partial B_{2r}(z_0)} |a_j(z)| \qquad \square$$

$$\sup_{z\in K}\left|\frac{f_j(z)}{g_j(z)}-\frac{f(z)}{g(z)}\right|$$

$$\leq \sup_{z\in K}\left|\frac{f_j(z)-f(z)}{g_j(z)}\right| + \sup_{z\in K}\left|f(z)\left(\frac{1}{g_j(z)}-\frac{1}{g(z)}\right)\right|$$

$$\leq \frac{2}{C}\underbrace{\sup_{z\in K}\left|f_j(z)-f(z)\right|}_{\to 0} + \overbrace{\sup_{z\in K}\left|f(z)\right|}^{<\infty}\ \sup_{z\in K}\left|\frac{1}{g_j(z)}-\frac{1}{g(z)}\right|$$

Note that the function $z \longmapsto 1/z$ is Lipschitz cts away from $z=0$. Hence $\forall j \geq j_0$:

$$\sup_{z\in K}\left|\frac{1}{g_j(z)}-\frac{1}{g(z)}\right| \leq L\underbrace{\sup_{z\in K}\left|g_j(z)-g(z)\right|}_{\to 0}$$

by loc unif cvg.

$\square$